

High fidelity quantum gates of trapped ions mediated by a dissipative bus mode

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We describe the optimal realization of entangling quantum gates for trapped ions mediated by a dissipative bus mode. With suitably shaped control pulses one can substantially decrease ion-phonon entanglement while maintaining the mediated interaction.

The realization of quantum gates is the fundamental building block in the exploitation of quantum mechanical systems for purposes like the execution of quantum algorithms, digital quantum simulations, quantum teleportation or precision sensing [1–5]. Some proof-of-principle experiments can be performed with mediocre quantum gates, but harnessing the true potential of quantum mechanical systems requires high-fidelity gates. In the last years one could therefore witness dedicated efforts towards the increase of gate fidelities. Single qubit gates for trapped ions have been implemented with infidelities reaching 10^{-6} [6]; but given the increased complexity, fidelities for entangling gates for two ions are substantially lower, in the range of 10^{-2} [7].

The potential of trapped ions

for applications of quantum information theory has been demonstrated abundantly [8–10], but the limited fidelities of entangling gates still pose an obstacle to fault tolerance. Despite the fact that qubits can be encoded in highly stable hyperfine, dressed or clock states [11, 12] with very long coherence times, entangling gates are limited by the comparatively short coherence times of the collective oscillation that is needed to mediate an interaction between the otherwise non-interacting qubit degrees of freedom. Since the implementation of a gate necessarily implies some entanglement between the qubits and this bus mode, the decoherence of the latter unavoidably has a detrimental impact on the former.

Substantial progress towards the goal of making qubits independent of the bus mode decoherence was made in terms of the Mølmer-Sørensen (MS) gate [13]. It is independent of the state of the bus mode; that is, it is insensitive to any type of decoherence prior to the gate operation. This feature has helped substantially towards the experimental implementation of two-qubit and multi-qubit gates [14–16], but sensitivity to bus mode decoherence during the gate operation is still a limiting factor. As we show here, this sensitivity can be reduced substantially through suitably chosen polychromatic driving.

The MS gate is induced by two driving fields that are slightly detuned with respect to the red and blue sideband transitions respectively. The corresponding Hamiltonian reads

$$\mathcal{H}(t) = (\Upsilon(t) a + \Upsilon^*(t) a^\dagger) \mathcal{S}_x,$$

in terms of the creation and annihilation operators a^\dagger and a of phonons in the bus mode and the collective operator

$\mathcal{S}_x = \sum_j \sigma_x^{(j)}$ where $\sigma_x^{(j)}$ induces transitions between the two qubit states of the j -th ion. The time-dependent amplitude $\Upsilon(t)$ typically equals $\eta\Omega \exp(i\delta t)$ with the Lamb-Dicke parameter η , the Rabi frequency Ω and the detuning δ [13], but, in the following, we will consider more general time dependencies.

The propagator induced by $\mathcal{H}(t)$ reads

$$\mathcal{U}_K = \exp(-i((f(t)a + f^*(t)a^\dagger)\mathcal{S}_x - g(t)\mathcal{S}_x^2)) \quad (1)$$

with $f(t) = \int_0^t dt' \Upsilon(t')$ and $g(t) = \Im \int_0^t dt' \Upsilon(t') f^*(t')$.

At instances at which f vanishes, this describes the dynamics induced by the spin-spin-type interaction \mathcal{S}_x^2 independently of the motional state of the bus mode. This feature permits implementation of multi-qubit quantum gates even with incoherent, thermal states, what has resulted in experimental implementations with high gate fidelities [7, 17, 18]. In the quest for gates that permit the realization of fault tolerant quantum computation, however, incoherent processes during the gate operation are a limiting factor [19]. The qubits and motional degrees of freedom get correlated (as accounted for by the term $(f(t)a + f^*(t)a^\dagger)\mathcal{S}_x$ in Eq. (1)) so that dissipation of the ions' motion affects the coherence of the qubits. Predominant effects are thermalization and dephasing that can be modeled with a Master equation characterized by a generator $\mathcal{L}[\circ] = -i[\mathcal{H}(t), \circ] + \sum_j \gamma_j \mathcal{D}_{E_j}[\circ]$ [20]. The first term $-i[\mathcal{H}(t), \circ]$ describes coherent dynamics and the dissipator is comprised of the terms $\mathcal{D}_{\hat{O}}[\circ] = \hat{O} \circ \hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger \hat{O}, \circ\}$ with operators $E_1 = a$ and $E_2 = a^\dagger$ for thermalization and $E_3 = a^\dagger a = \hat{n}$ for dephasing.

In the following we consider all dynamics in the frame defined by \mathcal{U}_K in which a perfect, entangling gate is described by the identity \mathbb{I} , and the Master equation becomes a purely dissipative equation with a time-dependent dissipator $\tilde{\mathcal{L}}[\circ] = \sum_j \gamma_j \mathcal{D}_{\tilde{E}_j(t)}[\circ]$ with

$$\begin{aligned} \tilde{E}_1 &= a - if^*(t)\mathcal{S}_x, \\ \tilde{E}_2 &= a^\dagger + if(t)\mathcal{S}_x, \\ \tilde{E}_3 &= \hat{n} + i(af(t) - a^\dagger f^*(t))\mathcal{S}_x + |f(t)|^2 \mathcal{S}_x^2. \end{aligned}$$

The Lindblad operators \tilde{E}_j directly affect both qubits and motional degrees of freedom, which is the formal manifestation of the fact that motional decoherence affects the qubits during gate operation. In state of the art experiments the decay rates are sufficiently small

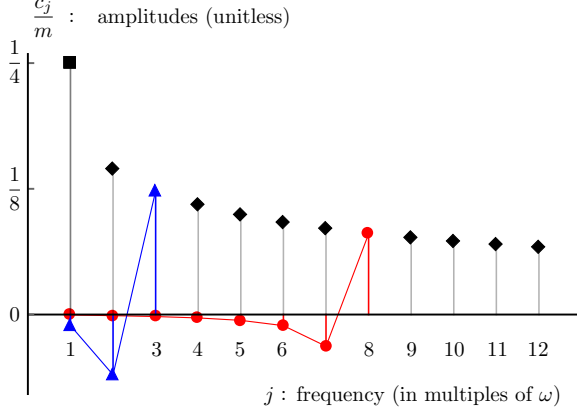


FIG. 1. (color online) The blue triangles (red circles) depict the amplitudes c_j/m of the optimal pulse (Eq. (3)) with $m = 3$ (8) frequency components – the line connecting the symbols is only to guide the eye. The amplitude of the highest frequency component is always dominant and phase shifted as compared to all other amplitudes. The filled square depicts the amplitude for a monochromatic pulse and the diamonds depict the amplitude of the highest (and dominant) frequency component for c_m/m for m ranging from 2 to 12. The conservative comparison $m\omega = \text{const.}$ implies that gate duration increases with m and the amplitudes c_j/m get lower with increasing m as depicted here.

such that $\gamma_j T \ll 1$, where T is the gate duration. The propagator induced by the time-dependent generator $\tilde{\mathcal{L}}$ can thus be approximated perturbatively as $\tilde{\mathcal{V}}(t) \simeq \mathbb{I} + \int_0^t dt' \tilde{\mathcal{L}}_{t'}$, and the infidelity \mathcal{I} of the gate operation can be characterized (in leading order) by the magnitude of the map

$$\Xi(o) = \text{Tr}_{\mathcal{B}} \int_0^T dt' \tilde{\mathcal{L}}_{t'}[o \otimes \varrho_{\mathcal{B}}], \quad (2)$$

where $\text{Tr}_{\mathcal{B}}$ denotes the trace over the bus mode with the density operator $\varrho_{\mathcal{B}}$. In practice, it is defined as $\mathcal{I} = \|\zeta\|^2 = \text{Tr} \zeta^\dagger \zeta$ in terms of the norm of the matrix ζ , with elements related to Ξ via $\Xi(o) = \sum_{ij} \zeta_{ij} \xi_i \circ \xi_j^\dagger$. The N -qubit operators utilized in this expression are defined as $\xi_0 = \mathbb{I}$ and

$$\xi_i = \binom{N}{i}^{-\frac{1}{2}} \sum_{1 \leq j_1 < \dots < j_i \leq N} \sigma_x^{(j_1)} \dots \sigma_x^{(j_i)}$$

for $i = 1, 2, 3, 4$.

The gate infidelity depends on the state of the bus mode via the properties $\langle \hat{n} \rangle$, $\langle a \rangle$, $\langle a^2 \rangle$ as shown in more detail in the appendix. Since, in practice, a control scheme is desirable that is independent of the state of the bus mode and the exact form of its dissipation, we will refrain from optimizing the gate fidelity for very specific situations, but rather target solutions that yield good performance largely independent of bus mode properties.

To this end, it is helpful to realize that ζ depends on the pulse $\Upsilon(t)$ via the functions $\langle f \rangle$, $\langle |f|^2 f \rangle$, $\langle |f|^2 \rangle$ and $\langle |f|^4 \rangle$, where the symbol $\langle \cdot \rangle = \int_0^T dt \cdot$ denotes time-integration over the gate duration T .

As deviations from perfect gate operations result from qubit-phonon entanglement as characterized by $f(t)$ in Eq. (1) one would expect to achieve high fidelities by minimizing the deviations of $f(t)$ from 0. It is always possible to choose $\Upsilon(t)$ such that $\langle f \rangle$ and $\langle |f|^2 f \rangle$ vanish, but since $\langle |f|^2 \rangle$ and $\langle |f|^4 \rangle$ are averages over non-negative functions, they will always be non-negative for a non-vanishing pulse $\Upsilon(t)$. We found that a large improvement is obtained if $\langle f \rangle$ vanishes, but that the additional requirement that also $\langle |f|^2 f \rangle$ vanish yields hardly further improvement. In fact, the second most relevant aspect is the minimization of $\langle |f|^2 \rangle$, and we observed better gate performances with a minimization of $\langle |f|^2 \rangle$ under the constraint $\langle f \rangle = 0$ rather than the two constraints $\langle f \rangle = \langle |f|^2 f \rangle = 0$. Since the pulse $\Upsilon(t)$ obtained in this fashion also typically yields minimal values of $\langle |f|^4 \rangle$ this permits to identify driving patterns that achieve optimal gates, irrespective of the bus mode properties.

With the explicit parametrization

$$\Upsilon_p = \sum_{j=1}^m c_j \omega \exp(ij\omega t)$$

in terms of the fundamental frequency ω and unitless amplitudes c_j , the condition for $\mathcal{U}_{\mathcal{K}}(T)$ to be a maximally entangling gate $\exp(-i\frac{\pi}{8} S_x^2)$ reads $g_T = \sum_j |c_j|^2 / j = 1/16$, and the optimization to be solved reads

$$\min_{\{c_j\}} \left(\sum_{j=1}^m \frac{|c_j|^2}{j^2} \left| \sum_{j=1}^m \frac{|c_j|^2}{j} = \frac{1}{16}, \sum_{j=1}^m \frac{c_j}{j} = 0 \right. \right).$$

The optimal solution [21], that, in particular can be chosen real, reads

$$c_j = \frac{jb}{1-j\lambda} \quad \text{and} \quad b = \frac{-1}{4m} \left(\sum_{j=1}^m \frac{j}{(1-j\lambda)^2} \right)^{-\frac{1}{2}}, \quad (3)$$

where λ is the smallest root of the equation $\sum_j (1-j\lambda)^{-1} = 0$ [22]. The exemplary cases of $m = 3$ and $m = 8$ are depicted in Fig. (1); independently of m , there is the general pattern that the absolute values of the amplitudes grow with increasing frequency and that there is a phase shift of π between the amplitude of the highest frequency and all other amplitudes. With this solution, the most off-resonant process yields the dominant contribution to the gate performance, and the phase shift of π between the dominant amplitude and all other amplitudes results in destructive interference that suppresses undesired qubit-phonon entanglement.

When comparing monochromatic and polychromatic driving one needs to bear in mind that the performance

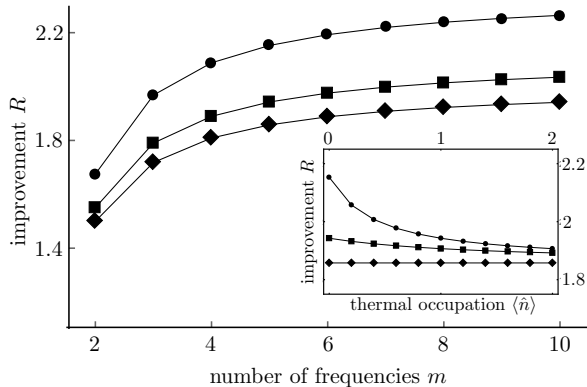


FIG. 2. Improvement R of gate fidelities as number of the frequencies in an optimized polychromatic pulse for a bus mode initially in the ground state. Circles correspond to vanishing thermalization and finite dephasing $\gamma_+ = \gamma_- = 0$, $\gamma_d > 0$, squares correspond to finite thermalization and dephasing with equal rates $\gamma_+ = \gamma_- = \gamma_d > 0$ and diamonds correspond to finite thermalization and vanishing dephasing $\gamma_+ = \gamma_- > 0$, $\gamma_d = 0$ (the lines guide the eye only). The inset depicts the dependence of the improvement R on the excitation $\langle \hat{n} \rangle$ for a thermal state of the bus mode and a pulse of $m = 5$ frequencies.

can always be improved through an increase of the fundamental frequency ω . In the following comparison we will thus always consider the case that the detuning δ of the monochromatic pulse equals the highest frequency $m\omega$ of the polychromatic pulse, so that the polychromatic pulse contains no higher frequencies than the monochromatic pulse. Since it is optimal to perform a gate within on driving period [19] we will consider this for both the monochromatic and the polychromatic case.

Fig. 2 depicts the improvement R defined in terms of the ratio of the gate infidelities \mathcal{I} for an optimized pulse and a monochromatic pulse. Circles, squares and diamonds correspond to different environment models ($\gamma_+ = \gamma_- = 0$, $\gamma_d > 0$, $\gamma_+ = \gamma_- = \gamma_d > 0$ and $\gamma_+ = \gamma_- > 0$, $\gamma_d = 0$), and the improvement is plotted a function of the number of frequencies m in the pulse Υ_p . The bus mode is assumed to be in its ground state. As one can see, a pulse with two frequencies (that permits to realize the condition $\langle f \rangle = 0$) yields an improvement by a factor around 1.5 to 1.7 depending on the bus mode dissipation. Increasing the number of frequencies enhances this improvement and values of R around 2 can be reached. The inset depicts the dependence of R on the mean phonon excitation $\langle \hat{n} \rangle$. As one can see in this exemplary case of $m = 5$, there is a slight drop of improvement with increasing $\langle \hat{n} \rangle$, but even in the limit of large excitations, improvements around 1.9 are reached.

One should point out that this comparison between

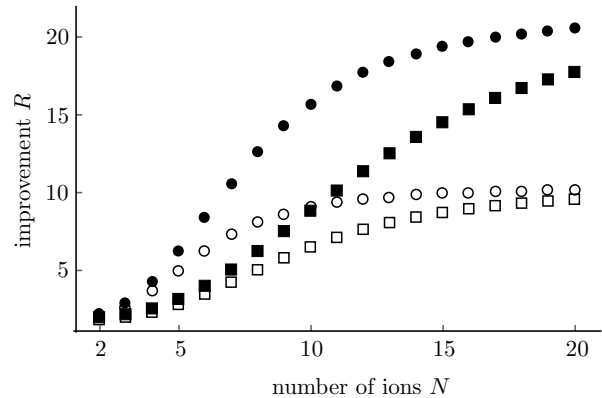


FIG. 3. Improvement R as function of the number of ions N for a pulse with $m = 3(5)$ frequencies depicted by empty(full) symbols. (circles correspond to $\gamma_+ = \gamma_- = 0$, $\gamma_d > 0$; squares correspond to $\gamma_+ = \gamma_- = \gamma_d > 0$) The improvement rapidly grows with N and the asymptotic improvement (for $N \rightarrow \infty$) grows with m .

monochromatic and polychromatic pulses is very conservative. Requiring the polychromatic pulse not to contain higher frequencies than the detuning δ of the monochromatic pulse implies that ω decreases inversely proportional to m and the gate duration is increased by a factor of m . This, however, indicates that the actual driving amplitudes $c_j\omega = c_j\delta/m$ are substantially lower than for a monochromatic pulse as depicted in Fig.(1). Indeed, even for a bichromatic pulse, the dominant amplitude is reduced by a factor of 1.7 as compared to the monochromatic case, and starting with $m = 4$ this reduction is larger than 2. Since this, in turn, reduces undesired off-resonant transitions, one can, in practice, work with a higher fundamental frequency ω than dictated by the constraint $m\omega = \delta$, and improve the gate fidelity even further.

As Fig. 3 shows, this improvement also grows quickly with the number of ions. For a pulse with only 3 frequencies R reaches values around 10, and a spectrally broader pulse with $m = 5$ yields values up to $R \simeq 20$. Only for an environment model that does not contain any pure phase loss (*i.e.* $\gamma_d = 0$), the improvement R is indeed independent of the number of ions. Such a situation, is however rather rare, since typically rates for energy-conserving dephasing processes are substantially larger than processes in which energy between system and environment is being exchanged. One would thus expect to benefit from rapid growth of R is most realistic situations.

The dependence of R on the number of ions N can be understood in terms of the elementary processes that contribute to a reduction of gate fidelities, and those are characterized in terms of the matrix ζ (defined in con-

text of Eq. (2)) whose norm defines the gate infidelity. As shown in the appendix, the rates for the occurrence of simultaneous phase-flips on four ions induced by dephasing of the bus mode scale with N^4 . Since most of these processes disappear for optimized pulses due to the constraint $\langle f \rangle = 0$, and the remaining contributions are being minimized, the improvement indeed grows with N .

A central task of quantum gates is the creation of highly entangled states, but given the highly non-linear dependence of entanglement on the underlying quantum state, the gate fidelity itself does not necessarily provide an accurate assessment of the achievable entanglement. Therefore, we characterize the achievable entanglement of a two-qubit gate in terms of entanglement of formation E [23]. Since this is done with a numerical exact simulation that is not limited to the regime of weak dissipation, it particularly permits to assess the gate performance in strongly dissipative systems. For $\gamma_+ = \gamma_- = \gamma_d = 10^{-3}\delta$ we found the ratios of the deviation from the ideal value of 1 to be 1.60 for a bichromatic pulse, 1.87 for a trichromatic pulse and 2.07 for a pulse with 6 frequencies. Increasing the dissipative rates by a factor of 10 slightly reduces these values to 1.51, 1.72 and 1.80, but underlines that substantial improvement is maintained even in the regime of rather strong dissipation.

Since the fidelities of entangling gates are the current bottleneck in the effort to reach fault tolerance, the increased robustness with respect to dissipation in the bus mode promises to advance control over trapped ions substantially. In particular in modern surface trap architectures that brings ions close to noisy trap electrodes [24, 25] this shall aide the optimal implementation of high fidelity coherent multi-qubit operations. Given that the situation of an interaction between highly coherent systems that is mediated via an intermediate system with inferior coherence properties is by no means limited to trapped ions, but is common in most hybrid architectures of designed quantum systems, the present control scheme promises to become a versatile and frequently employed tool of quantum control.

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APPENDIX

The functions $\langle f \rangle$, $\langle f^2 \rangle$, $\langle |f|^2 f \rangle$, $\langle |f|^2 \rangle$ and $\langle |f|^4 \rangle$ that characterize the gate fidelity, can be expressed as

$$\begin{aligned}\langle f \rangle &= \frac{2\pi i}{\omega} g_1, & \langle f^2 \rangle &= -\frac{2\pi}{\omega} g_1^2, \\ \langle |f|^2 f \rangle &= -i \frac{2\pi}{\omega} (g_2 - 2f_1 g_1 - |g_1|^2 g_1), \\ \langle |f|^2 \rangle &= \frac{2\pi}{\omega} (f_1 + |g_1|^2), \\ \langle |f|^4 \rangle &= \frac{2\pi}{w} (f_2 - 4\Re g_2 g_1^* + |g_1|^4 + 4f_1 |g_1|^2).\end{aligned}$$

in terms of the functions

$$\begin{aligned}g_1 &= \sum_{i=1}^m \frac{c_i}{i}, & g_2 &= \sum_{ijp=1}^m \frac{c_i c_j c_p^*}{ijp} \delta_{i+j,p}, \\ f_1 &= \sum_{i=1}^m \frac{|c_i|^2}{i^2}, & f_2 &= \sum_{ijpq=1}^m \frac{c_i c_j c_p^* c_q^*}{ijpq} \delta_{i+j,p+q},\end{aligned}$$

whith the (unitless) amplitudes that characterize the pulse $\Upsilon_p = \sum_{j=1}^m c_j \omega \exp(ij\omega t)$.

The matrix ζ whose norm yields the gate infidelity reads

$$\zeta = \frac{\pi}{\omega} (\zeta_0 + \zeta_1), \text{ with } \zeta_1 = \sum_{i=1}^6 \zeta_{1i}$$

and

$$\begin{aligned}\zeta_0 &= 2f_1(\gamma_+ + \gamma_- + (2\langle \hat{n} \rangle + 1)\gamma_d)M_1 \\ &\quad - 4f_2\gamma_d N(N-1)M_2 \\ \zeta_{11} &= 2|g_1|^2(\gamma_+ + \gamma_- + (2\langle \hat{n} \rangle + 1)\gamma_d)M_1, \\ \zeta_{12} &= 16|g_1|^2 f_1 \gamma_d N M_3, \\ \zeta_{13} &= 4\gamma_d(|g_1|^4 - 4\Re(g_1 g_2^*))N(N-1)M_2, \\ \zeta_{14} &= 4\Re(g_1^2 \langle a^2 \rangle) \gamma_d M_4, \\ \zeta_{15} &= 8\Re(g_1 |g_1|^2 \langle a \rangle) \gamma_d M_5, \\ \zeta_{16} &= 4\Im(g_1 \langle a \rangle)(\gamma_+ - \gamma_- + \gamma_d)M_6.\end{aligned}\tag{4}$$

The matrices M_i read

$$\begin{aligned}M_1 &= \begin{bmatrix} -N & 0 & -\sqrt{\Gamma_2} & 0 & 0 \\ 0 & 2\Gamma_1 & 0 & 0 & 0 \\ -\sqrt{\Gamma_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ M_2 &= \begin{bmatrix} 1 & 0 & \sqrt{\Gamma_2} & 0 & -3\sqrt{\Gamma_4} \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{\Gamma_2} & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -3\sqrt{\Gamma_4} & 0 & 0 & 0 & 0 \end{bmatrix}, \\ M_3 &= \begin{bmatrix} -(N-1) & 0 & -\sqrt{\Gamma_2}(N-1) & 0 & 3\sqrt{\Gamma_4} \\ 0 & 0 & 0 & 0 & 0 \\ -\sqrt{\Gamma_2}(N-1) & 0 & 2(N-1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3\sqrt{\Gamma_4} & 0 & 0 & 0 & 0 \end{bmatrix}, \\ M_4 &= \begin{bmatrix} 0 & 0 & \sqrt{\Gamma_2} & 0 & 0 \\ 0 & N & 0 & 0 & 0 \\ \sqrt{\Gamma_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ M_5 &= \begin{bmatrix} 0 & \sqrt{N(N-1)} & 0 & 3\sqrt{\Gamma_3} & 0 \\ \sqrt{N(N-1)} & 0 & -\sqrt{\Gamma_1\Gamma_2} & 0 & 0 \\ 0 & -\sqrt{\Gamma_1\Gamma_2} & 0 & 0 & 0 \\ 3\sqrt{\Gamma_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ M_6 &= \begin{bmatrix} 0 & -i\sqrt{N} & 0 & 0 & 0 \\ i\sqrt{N} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

with $\Gamma_p = \binom{N}{p}$. Since ζ_1 vanishes if g_1 (or, equivalently $\langle f \rangle$) vanishes, a large improvement in gate fidelity is achieved even with an only bi-chromatic pulse that permits to satisfy this constraint. ζ_0 can be reduced only through minimization of f_1 and f_2 , and thus will always be finite for a non-vanishing pulse.

Following Eq.(2), matrix elements $[M_i]_{pq}$ ($p, q = 0, \dots, 4$) are associated with processes in which p or q ions undergo a phase error. Since Γ_j is proportional to N^j for $N \gg 1$, the terms ζ_{12} and ζ_{13} that are proportional to $N^2 M_2$ and $N M_3$ are the dominant contributions to ζ_1 in the limit of many ions. Since these terms are proportional to γ_d , a particularly large improvement can be obtained in the presence of finite dephasing. Since however, also ζ_0 contains a term proportional to $N^2 M_2$, this improvement saturates to a finite value even in the limit $N \rightarrow \infty$.